7. RECAP: GOVERNING EQUATIONS

STRAIN-DISPLACEMENT

$$\underset{\sim}{\epsilon} = \frac{1}{2}\left(\underset{\sim}{\nabla} \otimes \underset{\sim}{u} + \underset{\sim}{u} \otimes \underset{\sim}{\nabla}\right) \qquad 7.1$$

in 1D: $\epsilon = \dfrac{\partial u}{\partial x}$ $\qquad$ 7.2

CONSTITUTIVE RELATIONS

$$\underset{\approx}{\sigma} = \underset{\approx\approx}{C}\, \underset{\approx}{\epsilon} \qquad 7.3$$

in 1D: $\sigma = E\,\epsilon$ $\qquad$ 7.4

EQUATIONS OF MOTION

$$\underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma} + \underset{\sim}{f} = \rho\, \underset{\sim}{a} \qquad 7.5$$

in 1D $\quad \dfrac{\partial \sigma}{\partial x} + f_x = \rho a_x$ $\qquad$ 7.6

AS AN ILLUSTRATION, LET'S FOCUS ON 1D:

$$\frac{\partial \sigma}{\partial x} + f = \rho a$$

7.4

$$\frac{\partial}{\partial x}(E\epsilon) + f = \rho a$$

7.2

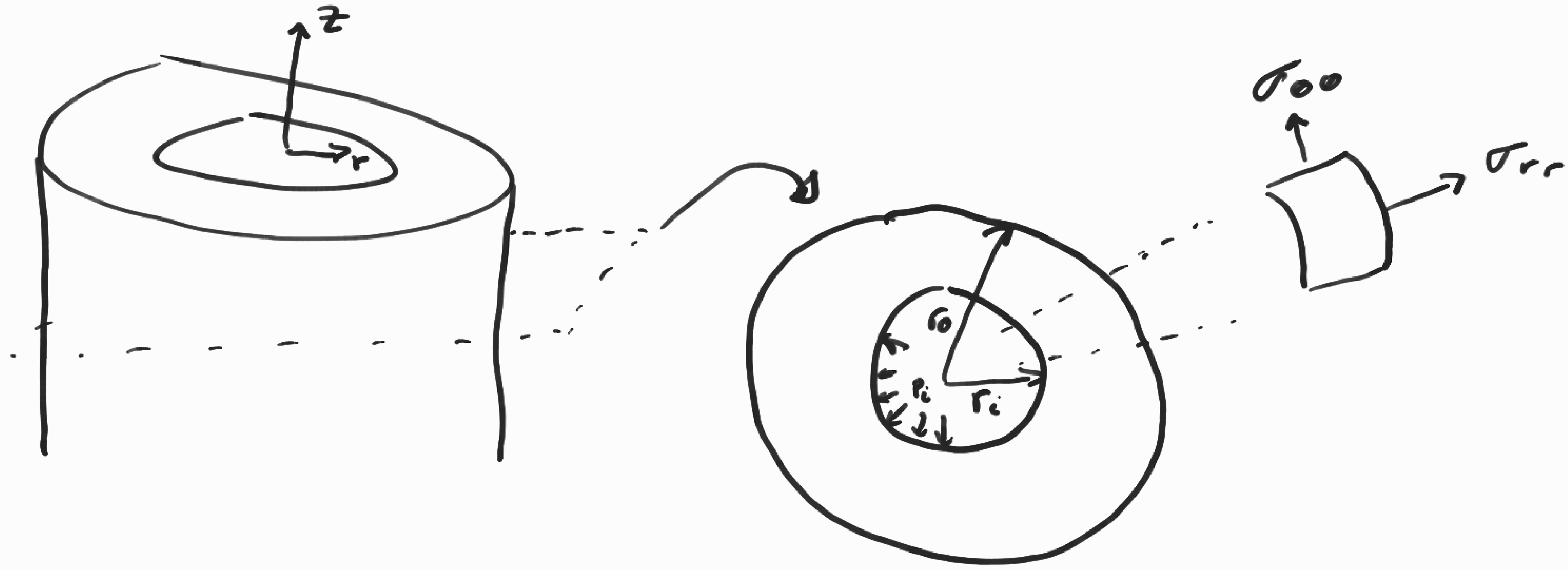
$$\frac{\partial}{\partial x}\left(E \frac{\partial u}{\partial x}\right) + f = \rho a$$

$$\boxed{E \frac{\partial u}{\partial x_2} + f = \rho \frac{\partial u}{\partial t^2}}$$

$\Rightarrow$ SOLID MECHANICS IS DIFFEQ!

8. STRESS ANALYSIS OF AN ARTERY

WE WILL MODEL AN ARTERY AS A THICK-WALLED CYLINDER INFLATED BY AN INTERNAL PRESSURE p_i. FIND THE DISPLACEMENT DUE TO p_i AND THE STRESS "DISTRIBUTION."



ASSUMPTIONS

1. LINEAR ELASTIC, ISOTROPIC
2. NO EXTERNAL LOADS, NO SHEAR

LET DISPLACEMENT $\underset{\sim}{u}(\underset{\sim}{x})$ BE DESCRIBED BY

$$\underset{\sim}{u} = u_r \underset{\sim}{e}_r + u_\theta \underset{\sim}{e}_\theta + u_z \underset{\sim}{e}_z \qquad (8.1)$$

BOUNDARY CONDITIONS

AT $r = r_i$, $\sigma_{rr} = -p_i$, $\sigma_{rz} = \sigma_{r\theta} = 0$

AT $r = r_0$, $\sigma_{rr} = \sigma_{rz} = \sigma_{r\theta} = 0$ $\qquad$ 8.2

SYMMETRY CONDITION

$$\frac{\partial}{\partial \theta} = 0$$ 8.3

MORE ASSUMPTIONS

1. $\sigma_{ij} = \sigma_{ij}(r)$, $\epsilon_{ij} = \epsilon_{ij}(r)$, $u_r = u_r(r)$
2. ALL SHEAR STRESS + STRAIN = 0 (r, θ, z = principal axes)
3. $\sigma_{zz} = 0$ throughout
4. IGNORE GRAVITY.

STRAIN - DISPLACEMENT IN CYLINDRICAL COOR.

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\epsilon_{\theta\theta} = \frac{u_r}{r} \qquad 8.4$$

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z}$$

CONSTITUTIVE EQ'S IN CYL.

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\epsilon_{rr} + \nu \epsilon_{\theta\theta}\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(\epsilon_{\theta\theta} + \nu \epsilon_{rr}\right) \qquad 8.5$$

EQUILIBRIUM EQ'S

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \qquad 8.6$$

TO BEGIN, SUBSTITUTE STRAIN-DISPL. 8.4 INTO

CONSTIT. REL. 8.5:

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\frac{du_r}{dr} + \nu\frac{u_r}{r}\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{\partial u}{\partial r}\right) \qquad 8.7a,b$$

FINALLY, SUB 8.7 INTO 8.6:

$$\boxed{\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0} \qquad 8.8$$

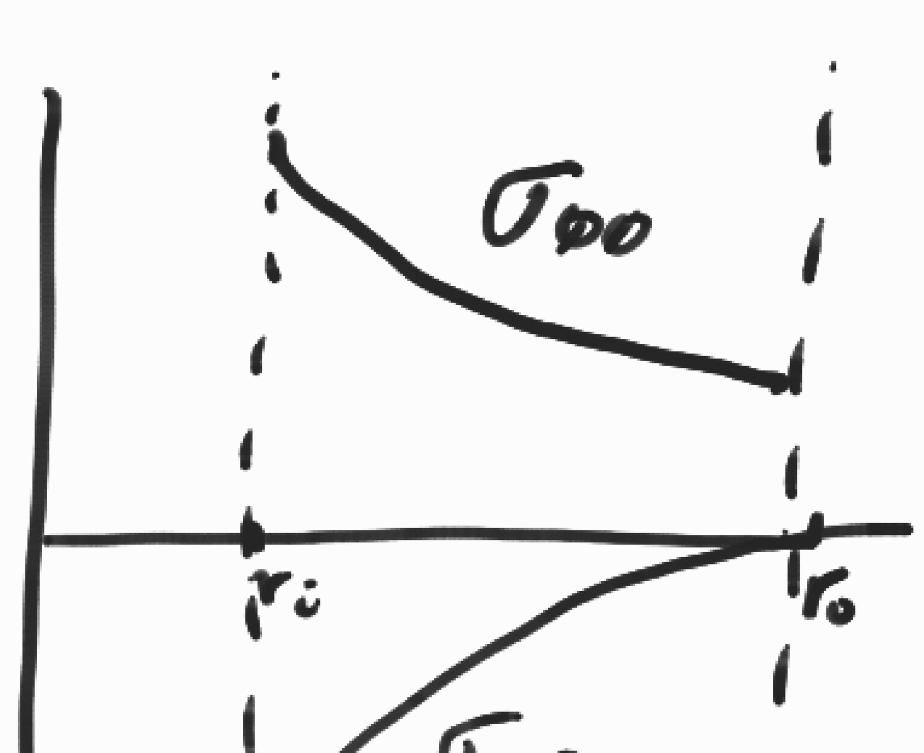
THE GENERAL SOLUTION FOR SUCH A PROBLEM IS

$$u_r(r) = C_1 r + \frac{C_2}{r} \qquad (8.9)$$

NOW APPLY B.C. TO SOLVE C_1 and C_2, GIVING EXACT SOLUTION

$$\sigma_{rr} = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right)$$

$$\sigma_{\theta\theta} = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r_2}\right)$$



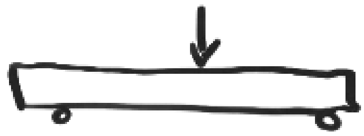

$-p_i$

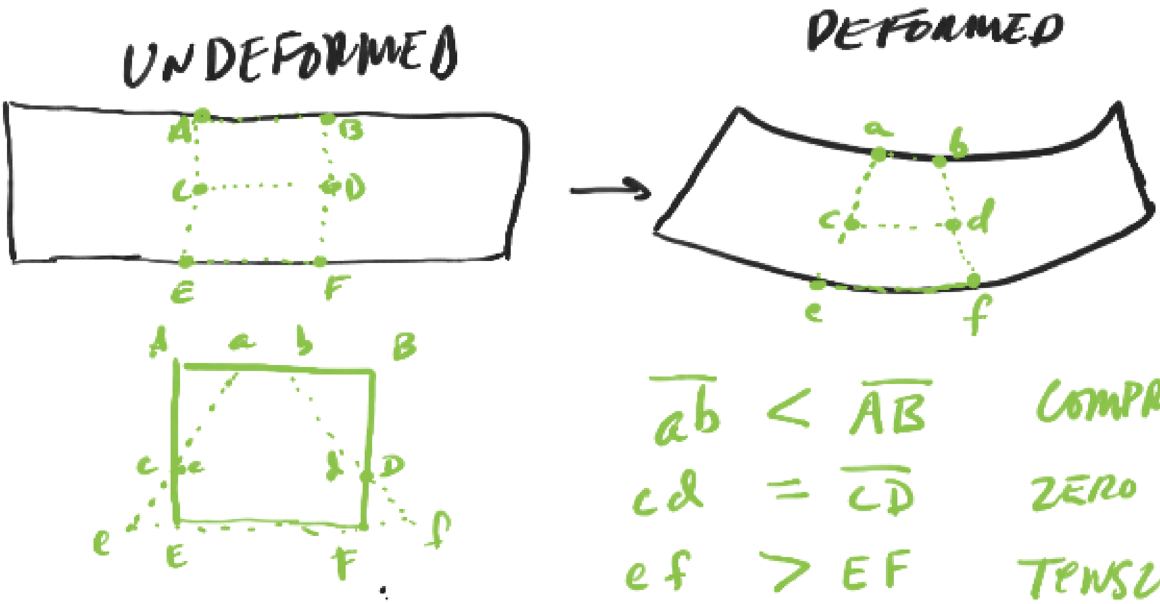
9. Beam bending

BEAM - A geometry IN WHICH ONE DIMENSION IS CONSIDERABLY LARGER THAN THE OTHER TWO.

A CANTILEVER BEAM

A SIMPLY SUPPORTED BEAM

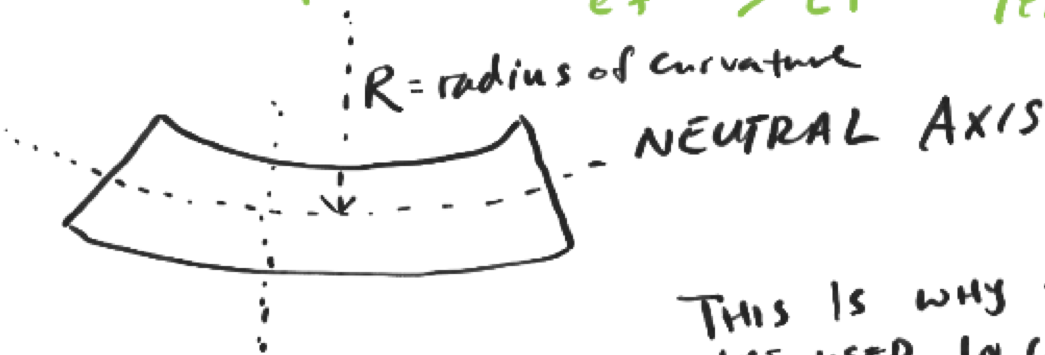


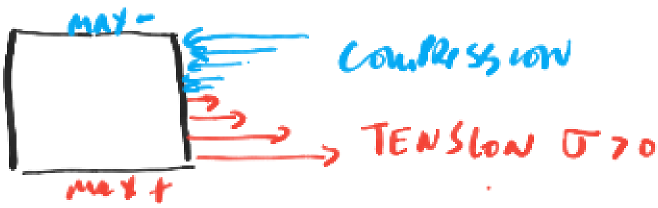


$\overline{ab} < \overline{AB}$ COMPRESSION

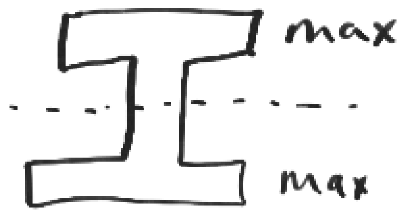
$\overline{cd} = \overline{CD}$ ZERO STRESS

$ef > EF$ TENSION





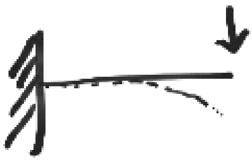
THIS IS WHY I BEAMS ARE USED IN CONSTRUCTION



NEUTRAL AXIS - LINE THROUGH BEAM LONG AXIS THA

EXPERIENCES NO STRESS OR STRAIN. FOR SYM. CROSS SECTION AND ISOTROPIC MAT'L WITH NO INITIAL CURVATURE, NEUTRAL AXIS IS THROUGH CENTROID.

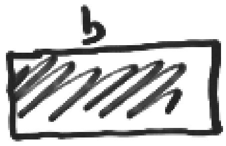
DEFLECTION - DISPLACEMENT OF BEAM $\perp$ to LONG AXIS



δ = DEFLECTION

SECOND MOMENT OF INERTIA - geometric property of 2D surface describing how points in that surface are distributed w.r.t. some Axis.

$$I = \frac{\pi r^4}{4} \qquad 9.1$$



$$I = \frac{ab^3}{12} \text{ or } \frac{a^3 b}{12} \qquad 9.2$$

ab

$$M = EI \frac{1}{R} \qquad 9.3$$

EI - FLEXURAL RIGIDITY

k_b

bending Stiffness

FOR A CANTILEVER

$\updownarrow \delta$

$$F = \frac{3EI}{L^3}\delta$$

FOR 3 POINT BENDING 

$F = \frac{48EI}{L^3}\delta$

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